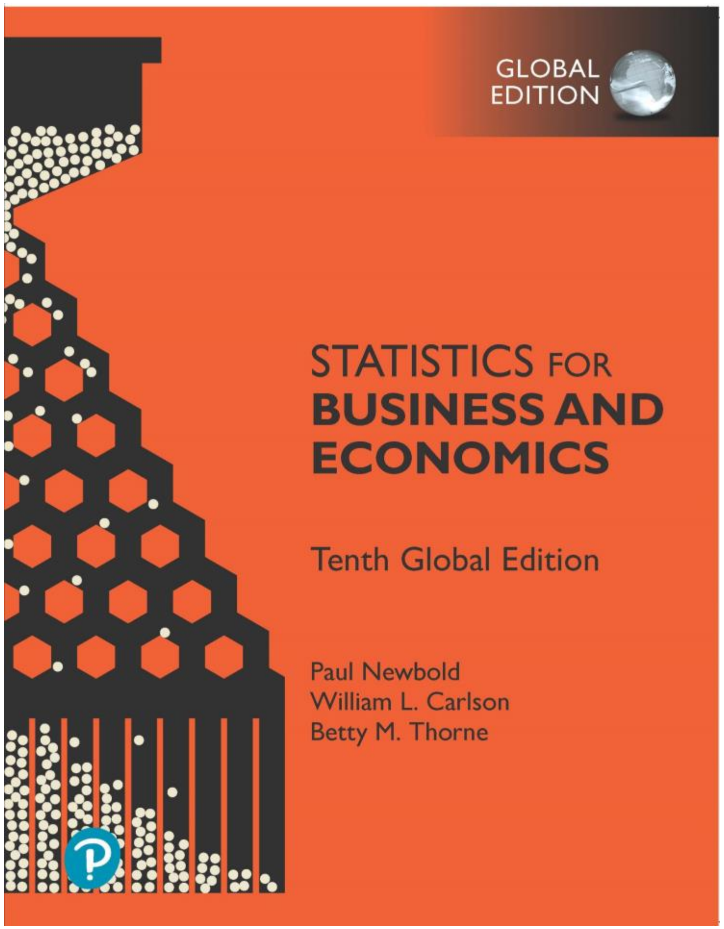
Statistics for Business and Economics

Tenth Edition, Global Edition



Chapter 6

Sampling and Sampling Distributions

Chapter Goals

After completing this chapter, you should be able to:

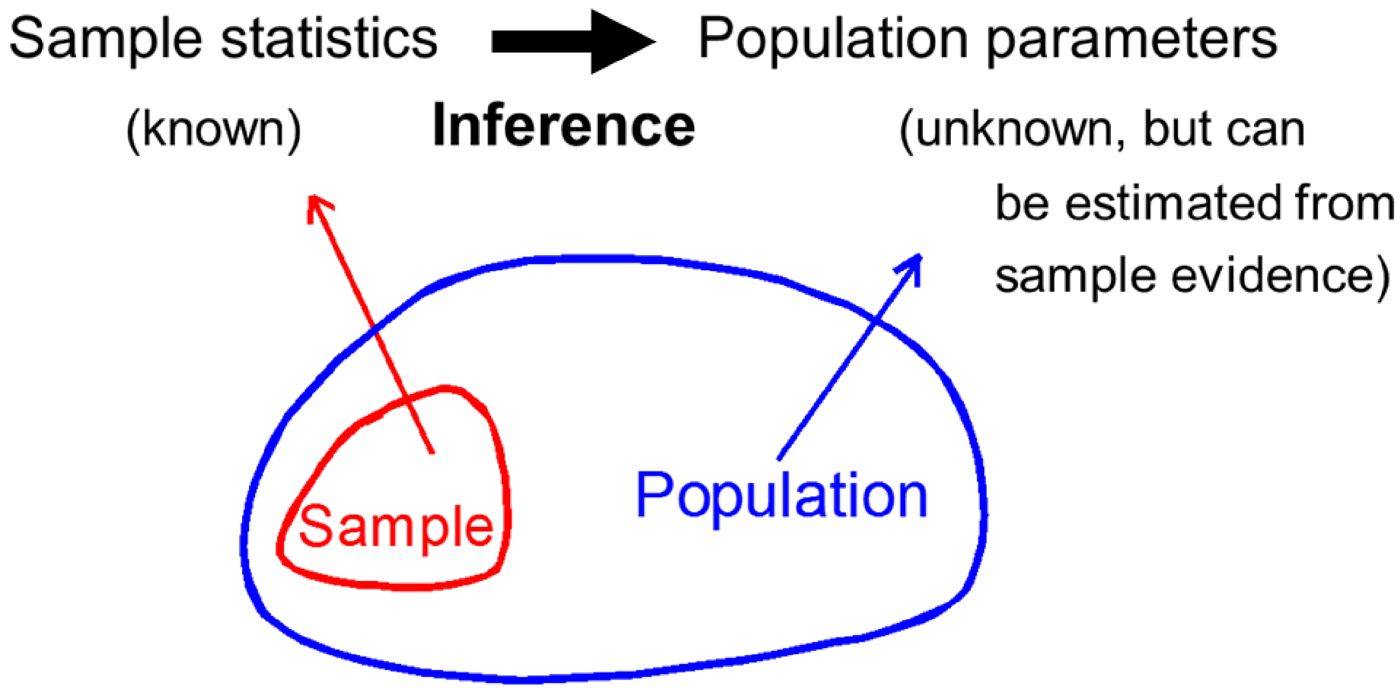
- Describe a simple random sample and why sampling is important
- Explain the difference between descriptive and inferential statistics
- Define the concept of a sampling distribution
- Determine the mean and standard deviation for the sampling distribution of the sample mean, $\overline{X}$
- Describe the Central Limit Theorem and its importance
- Determine the mean and standard deviation for the sampling distribution of the sample proportion, $\hat{p}$
- Describe sampling distributions of sample variances

Introduction

- **Descriptive statistics**
  - Collecting, presenting, and describing data
- **Inferential statistics**
  - Drawing conclusions and/or making decisions concerning a population based only on sample data

Inferential Statistics (1 of 2)

- Making statements about a population by examining sample results



Inferential Statistics (2 of 2)

Drawing conclusions and/or making decisions concerning a population based on sample results.

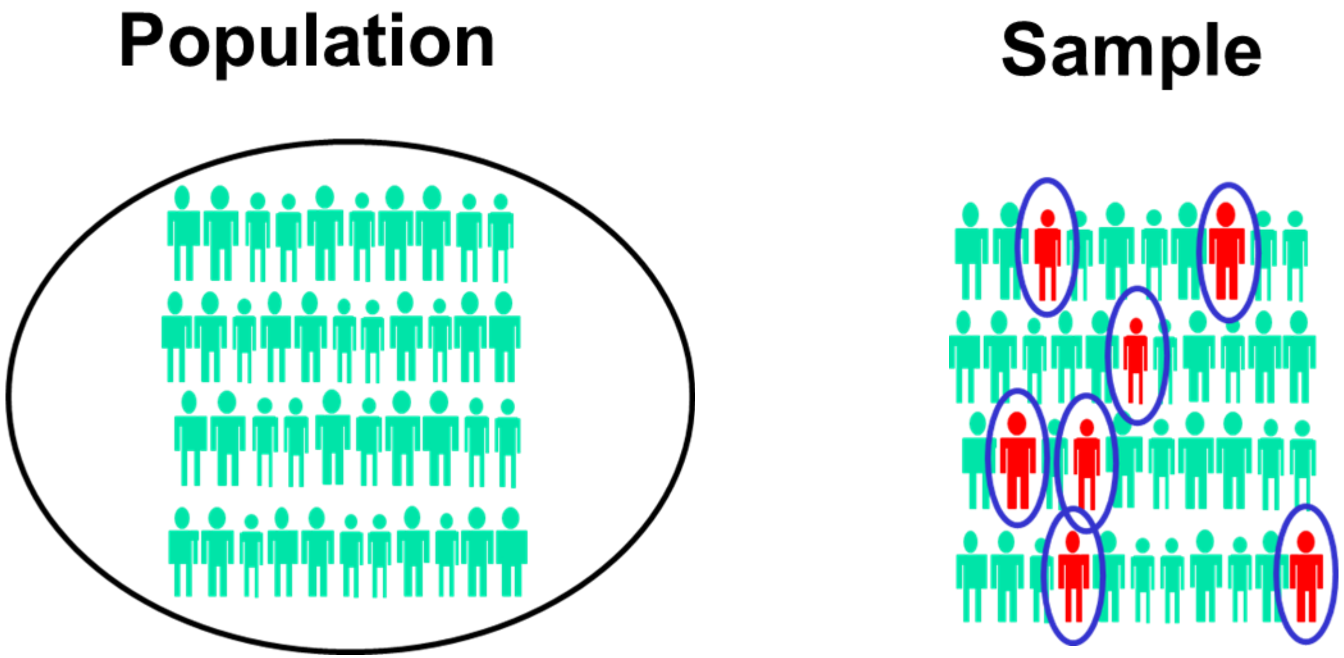
- **Estimation**
  - e.g., Estimate the population mean weight using the sample mean weight
- **Hypothesis Testing**
  - e.g., Use sample evidence to test the claim that the population mean weight is 120 pounds

Section 6.1 Sampling from a Population

- A **Population** is the set of all items or individuals of interest
  - Examples: All likely voters in the next election
    All parts produced today
    All sales receipts for November

- A **Sample** is a subset of the population
  - Examples: 1000 voters selected at random for interview
    A few parts selected for destructive testing
    Random receipts selected for audit

Population vs. Sample

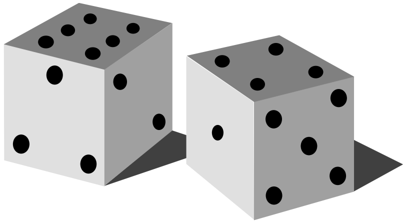


Why Sample?

- Less time consuming than a census
- Less costly to administer than a census
- It is possible to obtain statistical results of a sufficiently high precision based on samples.

Simple Random Sample

- Every object in the population has the same probability of being selected
- Objects are selected independently
- Samples can be obtained from a table of random numbers or computer random number generators



- A simple random sample is the ideal against which other sampling methods are compared

Sampling Distributions

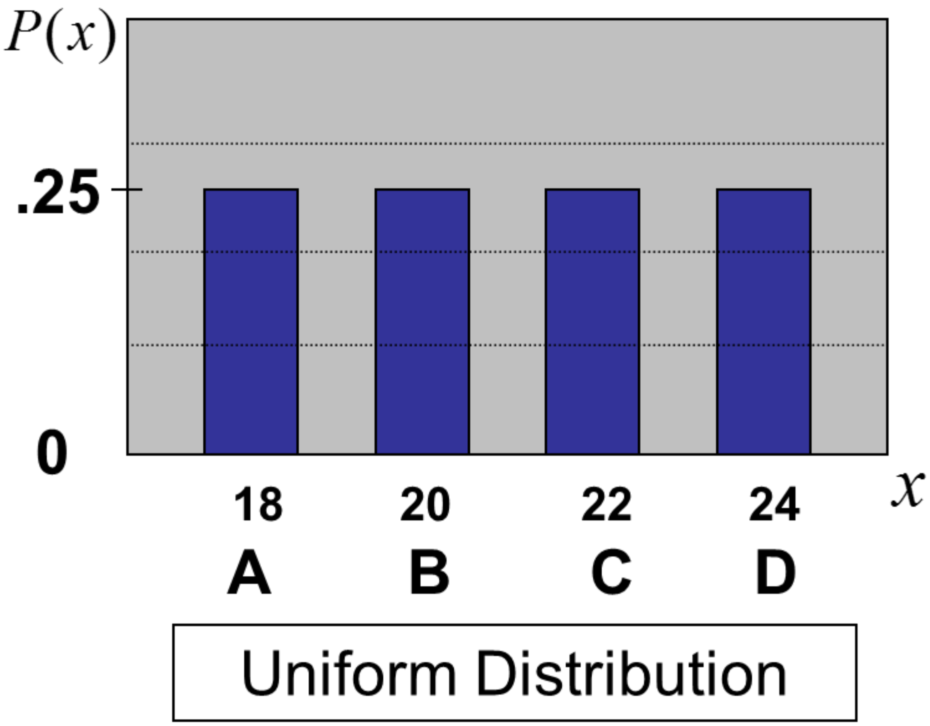
- A sampling distribution is a probability distribution of all of the possible values of a statistic for a given size sample selected from a population

Developing a Sampling Distribution (1 of 6)

- **Assume there is a population …**
- Population size N=4
- Random variable, X, is age of individuals
- Values of X: 18, 20, 22, 24 (years)

Developing a Sampling Distribution (2 of 6)

In this example the Population Distribution is uniform:



Developing a Sampling Distribution (3 of 6)

Now consider all possible samples of size $n = 2$

1st Obs	2nd Observation 18	20	22	24
18	18,18	18,20	18,22	18,24
20	20,18	20,20	20,22	20,24
22	22,18	22,20	22,22	22,24
24	24,18	24,20	24,22	24,24

16 possible samples (sampling with replacement)

16 Sample Means

1st Obs	2nd Observation 18	20	22	24
18	18	19	20	21
20	19	20	21	22
22	20	21	22	23
24	21	22	23	24

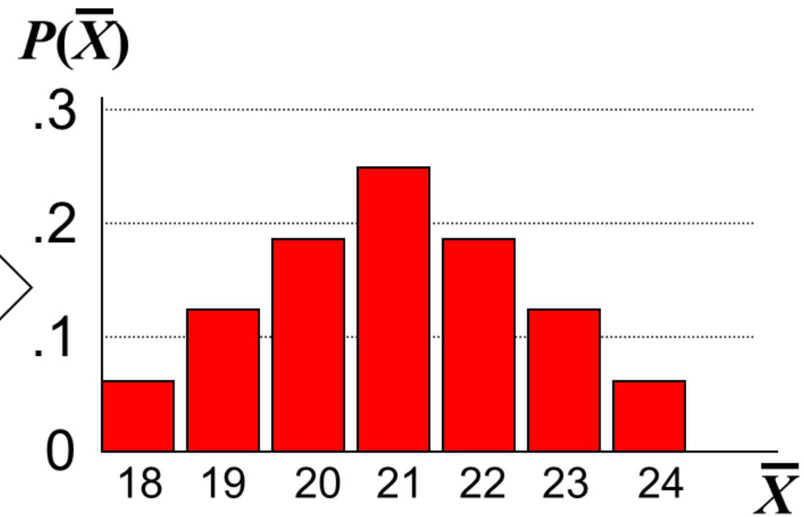
Developing a Sampling Distribution (4 of 6)

Sampling Distribution of All Sample Means

16 Sample Means

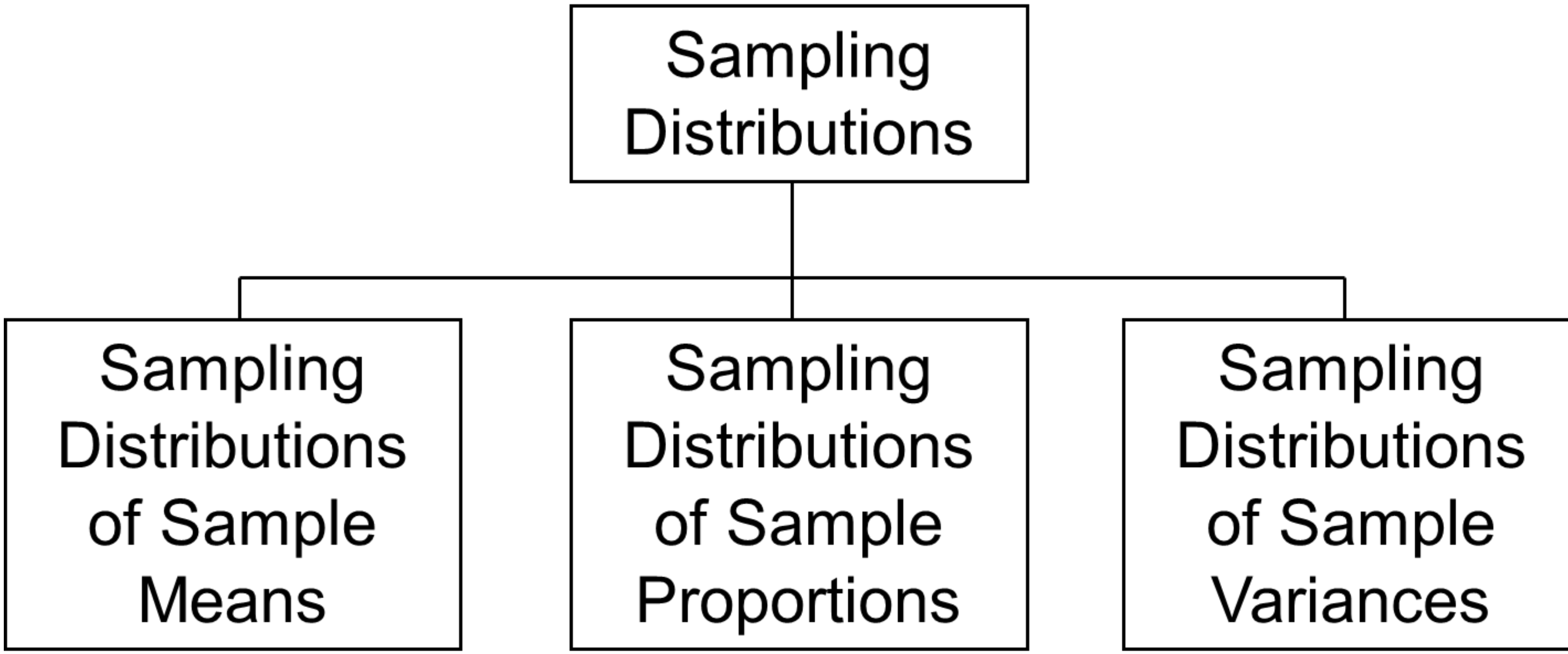
1st Obs	2nd Observation			
	18	20	22	24
18	18	19	20	21
20	19	20	21	22
22	20	21	22	23
24	21	22	23	24

Distribution of Sample Means

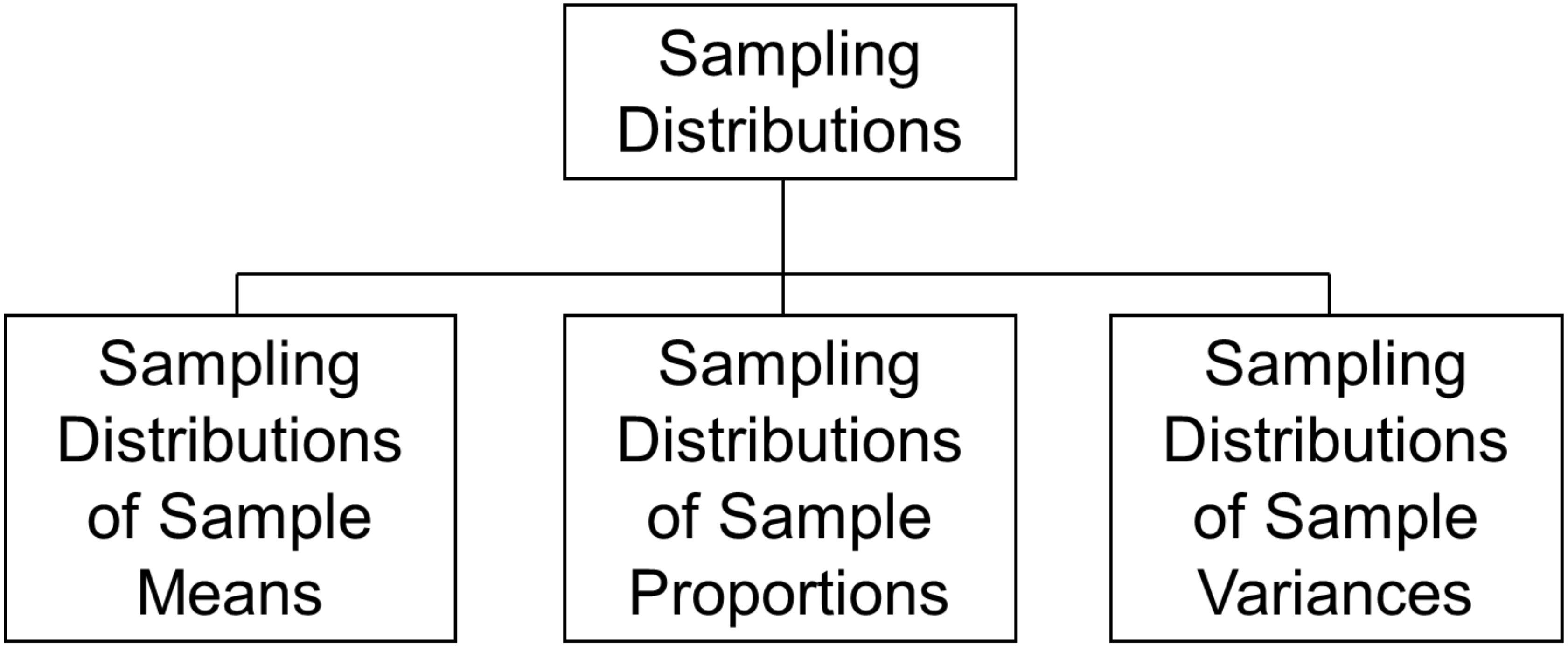


(no longer uniform)

Chapter Outline



Section 6.2 Sampling Distributions of Sample Means



Sample Mean

- Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population
- The sample mean value of these observations is defined as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Standard Error of the Mean

- Different samples of the same size from the same population will yield different sample means
- A measure of the variability in the mean from sample to sample is given by the Standard Error of the Mean:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

- Note that the standard error of the mean decreases as the sample size increases

Comparing the Population with Its Sampling Distribution

Population

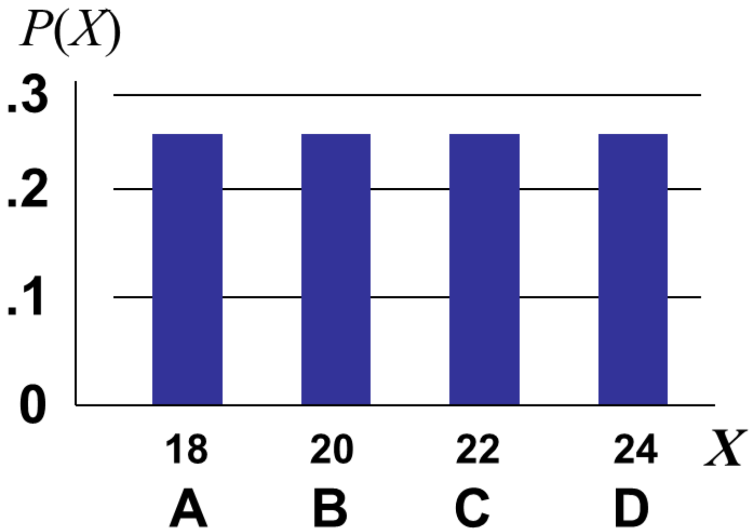
$$N = 4$$

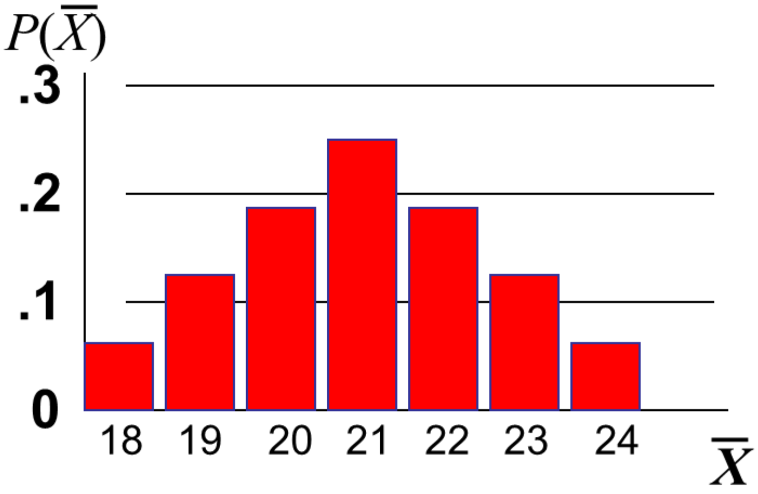
$$\mu = 21 \qquad \sigma = 2.236$$

Sample Means Distribution

$$n = 2$$

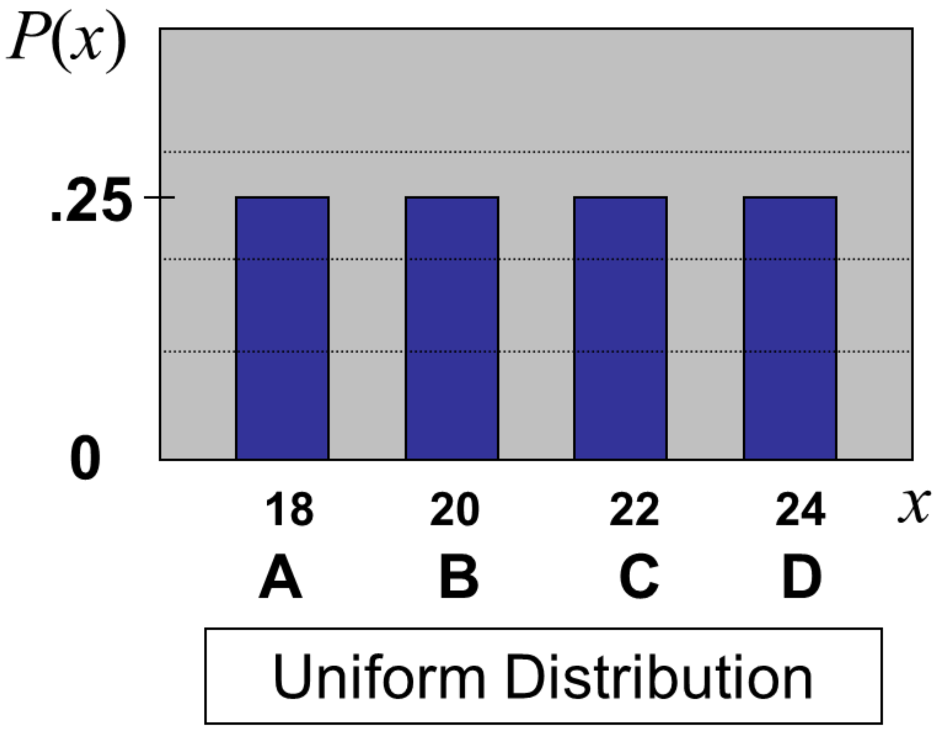
$$\mu_{\bar{X}} = 21 \qquad \sigma_{\bar{X}} = 1.58$$





Developing a Sampling Distribution (5 of 6)

Summary Measures for the Population Distribution:



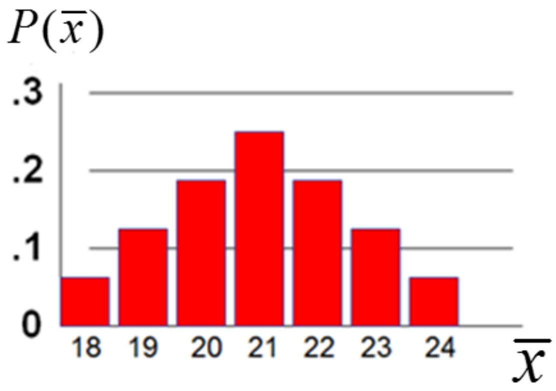
$$\mu = \frac{\sum X_i}{N}$$

$$= \frac{18 + 20 + 22 + 24}{4} = 21$$

$$\sigma = \sqrt{\frac{\sum (X_i - \mu)^2}{N}} = 2.236$$

Developing a Sampling Distribution (6 of 6)

Summary Measures of the Sampling Distribution:



$$E(\bar{X}) = \frac{\sum \bar{X}_i}{N} = \frac{18 + 19 + 21 + \cdots + 24}{16} = 21 = \mu$$

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum(\bar{X}_i - \mu)^2}{N}}$$

$$= \sqrt{\frac{(18-21)^2 + (19-21)^2 + \cdots + (24-21)^2}{16}} = 1.58$$

If Sample Values Are Not Independent

- If the sample size *n* is not a small fraction of the population size *N*, then individual sample members are not distributed independently of one another
- Thus, observations are not selected independently
- A finite population correction is made to account for this:

$$Var(\bar{X}) = \frac{\sigma^2}{n}\frac{N-n}{N-1} \quad \text{or} \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

The term $\frac{(N-n)}{(N-1)}$ is often called a **finite population correction factor**

If the Population Is Normal

- If a population is normal with mean μ and standard deviation σ, the sampling distribution of $\bar{X}$ is also normally distributed with

$$\mu_{\bar{X}} = \mu \quad \text{and} \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

- If the sample size *n* is not large relative to the population size *N*, then

$$\mu_{\bar{X}} = \mu \quad \text{and} \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

Standard Normal Distribution for the Sample Means

- *Z*-value for the sampling distribution of $\bar{X}$:

$$z = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

where: $\bar{X}$ = sample mean

μ = population mean

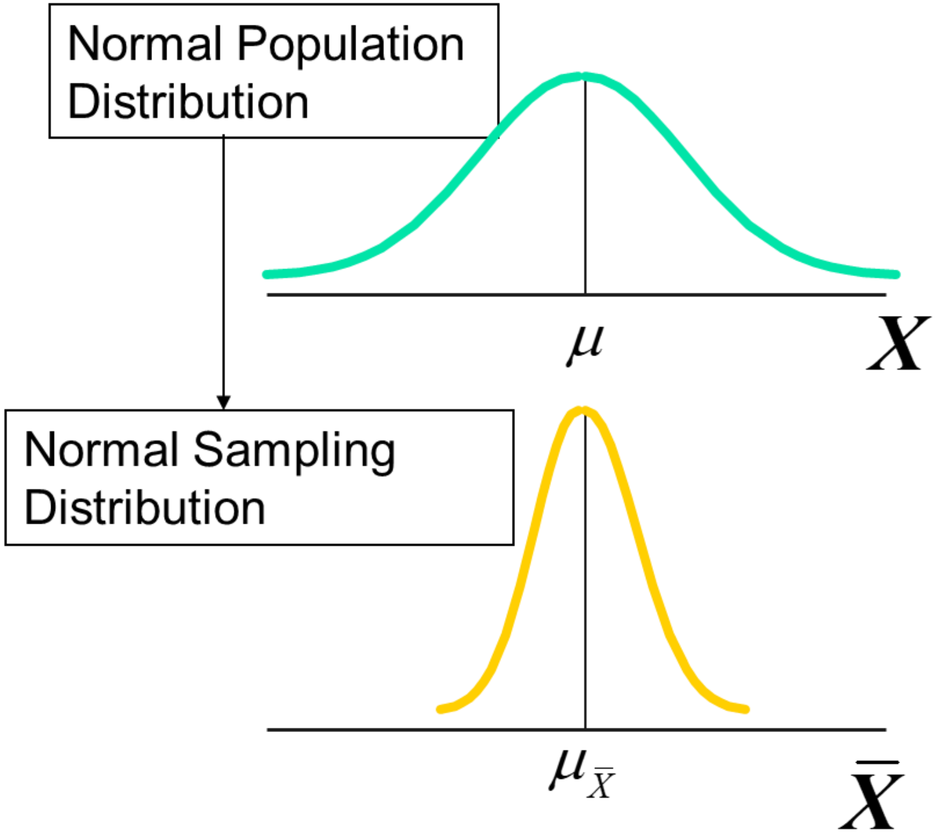
$\sigma_{\bar{X}}$ = standard error of the mean

Z is a standardized normal random variable with mean of 0 and a variance of 1

Sampling Distribution Properties (1 of 3)

$$E\left[\bar{X}\right] = \mu$$

(i.e. $\bar{X}$ is unbiased)

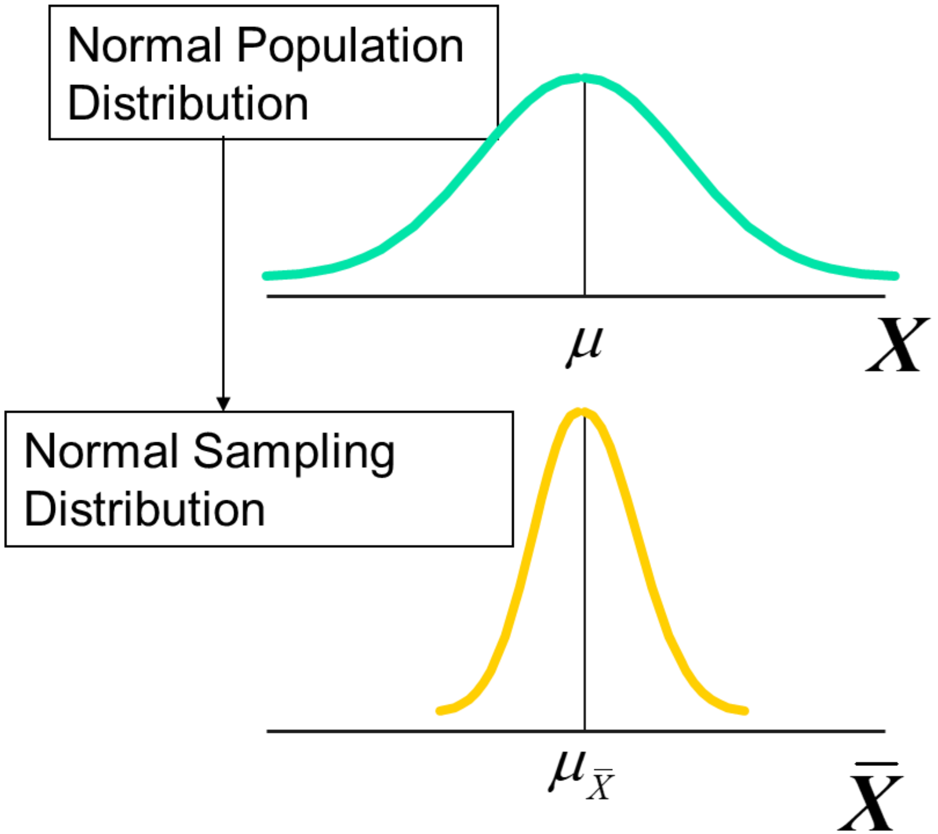


(both distributions have the same mean)

Sampling Distribution Properties (2 of 3)

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

(i.e. $\bar{X}$ is unbiased)

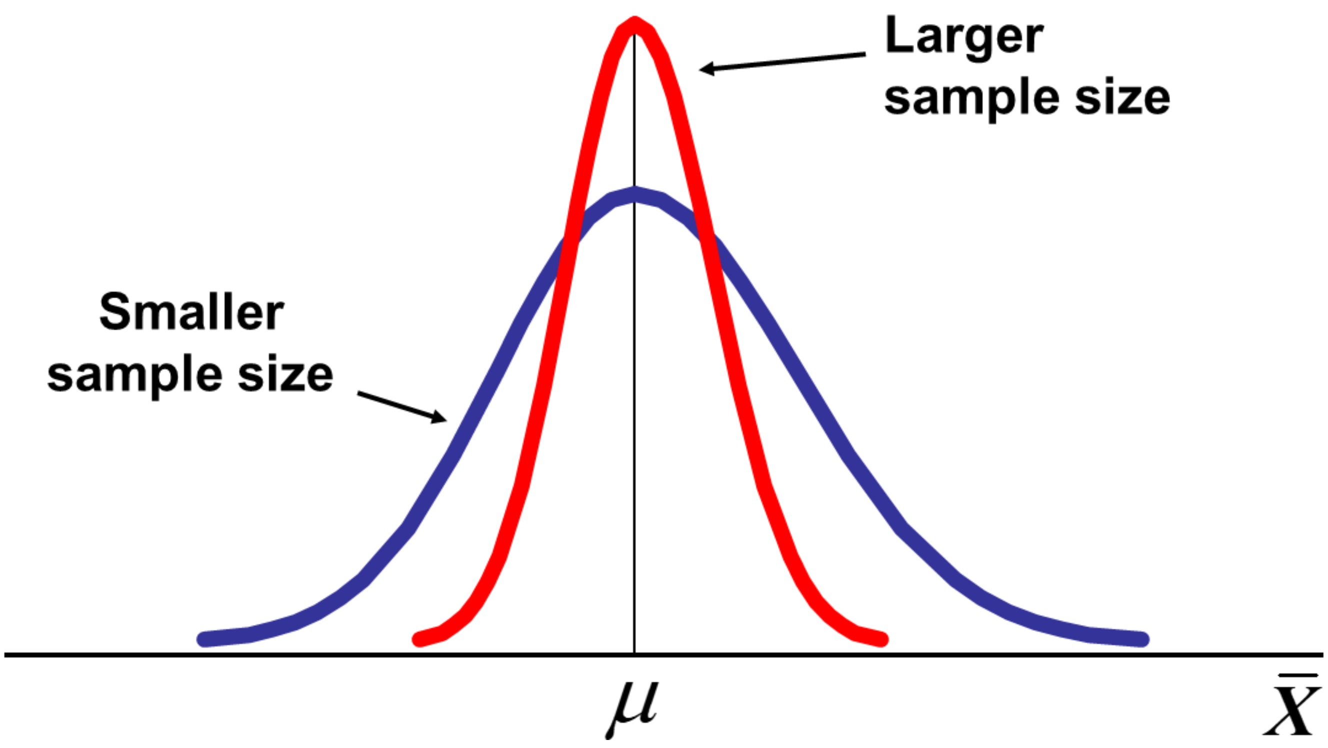


(the distribution of $\bar{X}$ has a reduced standard deviation)

Sampling Distribution Properties (3 of 3)

As *n* increases,

$\sigma_{\bar{X}}$ decreases



Central Limit Theorem (1 of 3)

- Even if the population is not normal,
- …sample means from the population will be approximately normal as long as the sample size is large enough.

Central Limit Theorem (2 of 3)

- Let $X_1, X_2, \ldots, X_n$ be a set of *n* independent random variables having identical distributions with mean μ, variance σ^2, and $\bar{X}$ as the mean of these random variables.
- As *n* becomes large, the central limit theorem states that the distribution of

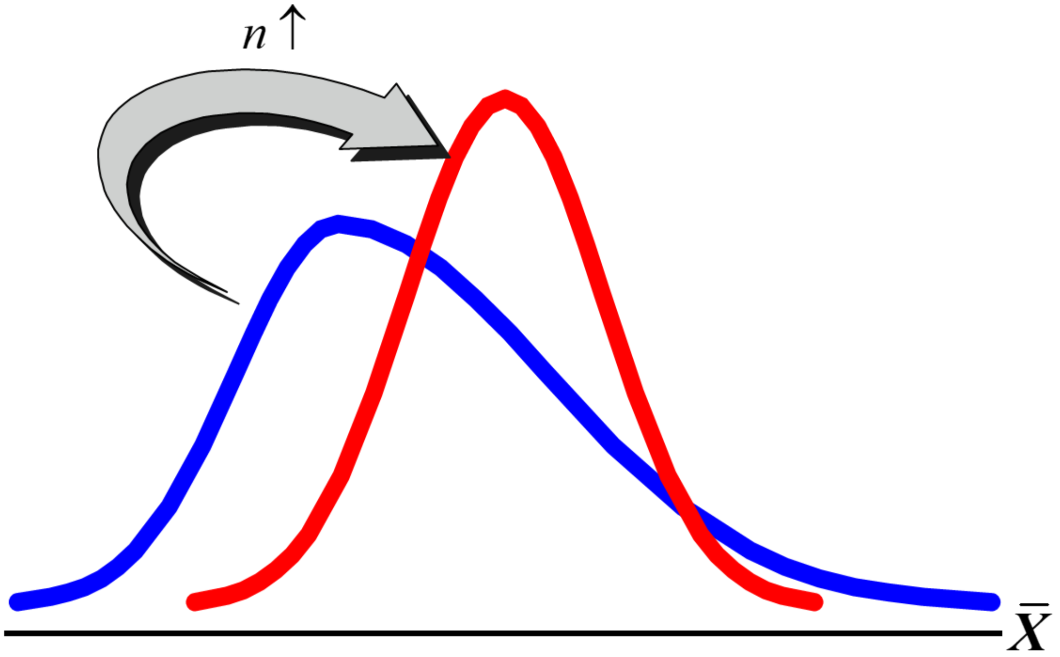
$$Z = \frac{\bar{X} - \mu_x}{\sigma_{\bar{X}}}$$

approaches the standard normal distribution

Central Limit Theorem (3 of 3)

As the sample size gets large enough…

the sampling distribution becomes almost normal regardless of shape of population



If the Population Is Not Normal

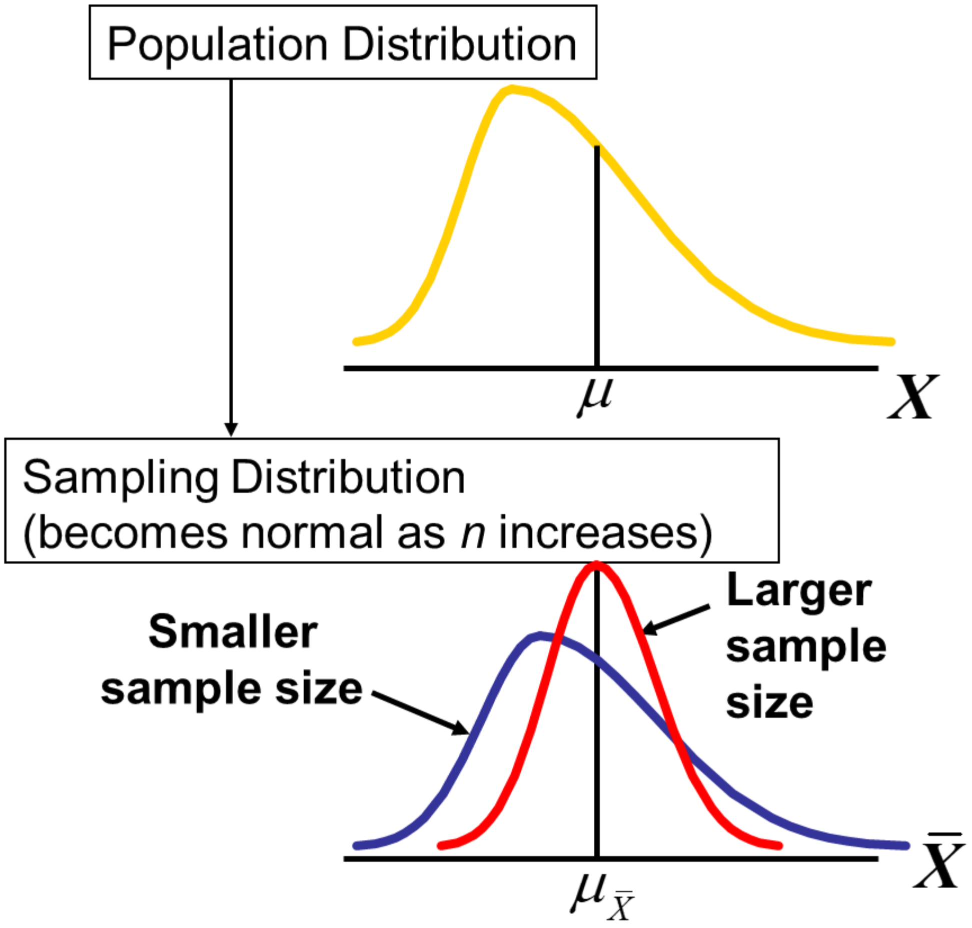
Sampling distribution properties:

Central Tendency

$$\mu_{\bar{X}} = \mu$$

Variation

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$



How Large Is Large Enough?

- For most distributions, $n > 25$ will give a sampling distribution that is nearly normal
- For normal population distributions, the sampling distribution of the mean is always normally distributed

Example 1 (1 of 3)

- Suppose a large population has mean $\mu = 8$ and standard deviation $\sigma = 3$. suppose a random sample of size n = 36 is selected.
- What is the probability that the sample mean is between 7.8 and 8.2?

Example 1 (2 of 3)

Solution:

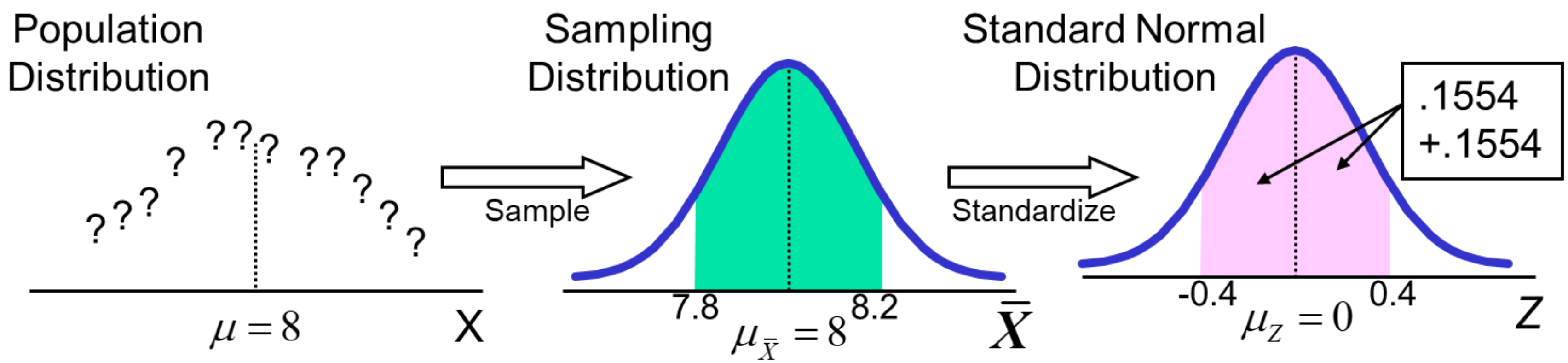
- Even if the population is not normally distributed, the central limit theorem can be used (*n* > 25)
- … so the sampling distribution of $\bar{X}$ is approximately normal
- … with mean $\mu_{\bar{X}} = 8$
- …and standard deviation $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{3}{\sqrt{36}} = 0.5$

Example 1 (3 of 3)

Solution: (continued):

$$P(7.8 < \mu_{\bar{x}} < 8.2) = P\left(\frac{7.8-8}{\frac{3}{\sqrt{36}}} < \frac{\mu_{\bar{x}} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{8.2-8}{\frac{3}{\sqrt{36}}} \right)$$

$$= P(-0.4 < Z < 0.4) = \boxed{0.3108}$$



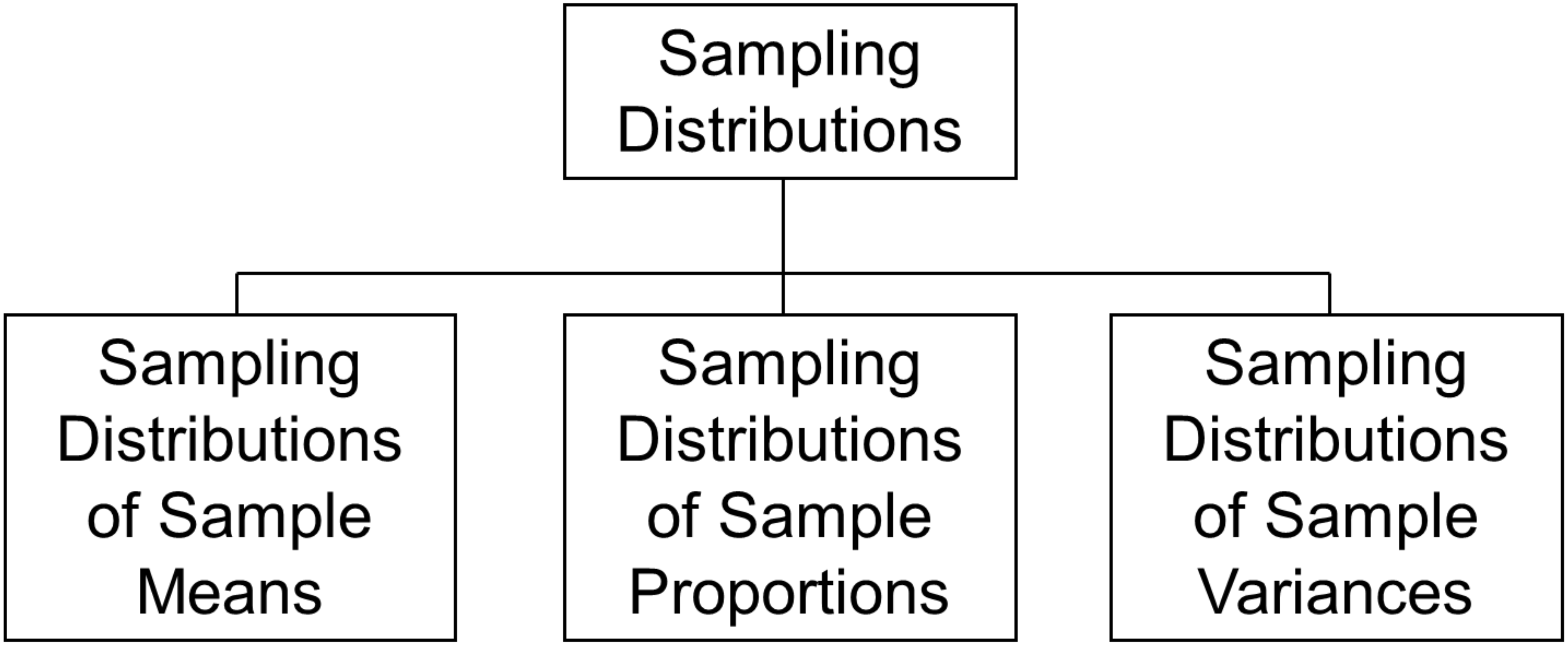
Acceptance Intervals

- Goal: determine a range within which sample means are likely to occur, given a population mean and variance
  - By the Central Limit Theorem, we know that the distribution of $\bar{X}$ is approximately normal if *n* is large enough, with mean μ and standard deviation $\sigma_{\bar{X}}$
  - Let $z_{\frac{\alpha}{2}}$ be the *z*-value that leaves area $\frac{\alpha}{2}$ in the upper tail of the normal distribution (i.e., the interval $-z_{\frac{\alpha}{2}}$ to $z_{\frac{\alpha}{2}}$ encloses probability $1-\alpha$)
  - Then

$$\mu \pm z_{\frac{\alpha}{2}} \sigma_{\bar{X}}$$

is the interval that includes $\bar{X}$ with probability $1-\alpha$

Section 6.3 Sampling Distributions of Sample Proportions



Sampling Distributions of Sample Proportions

P = the proportion of the population having some characteristic

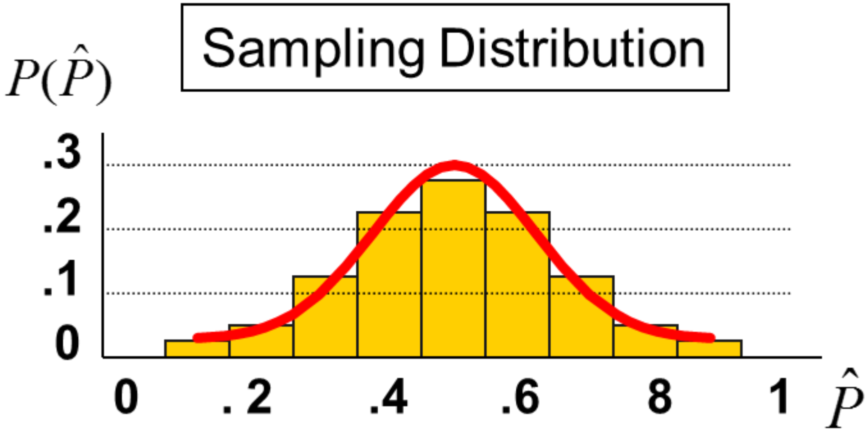
- Sample proportion $(\hat{p})$ provides an estimate of *P*:

$$\hat{p} = \frac{X}{n} = \frac{\text{number of items in the sample having the characteristic of interest}}{\text{sample size}}$$

- $0 \leq \hat{p} \leq 1$
- $\hat{p}$ has a binomial distribution, but can be approximated by a normal distribution when $nP(1-P) > 5$

Sampling Distribution of *p* Hat

- Normal approximation:



Properties: $E(\hat{p}) = P$ and $\sigma_{\hat{p}} = \sqrt{\frac{P(1-P)}{n}}$

(where *P* = population proportion)

Z-Value for Proportions

Standardize $\hat{p}$ to a *Z* value with the formula:

$$Z = \frac{\hat{p} - P}{\sigma_{\hat{p}}} = \frac{\hat{p} - P}{\sqrt{\frac{P(1-P)}{n}}}$$

Where the distribution of *Z* is a good approximation to the standard normal distribution if $nP(1-P) > 5$

Example 2 (1 of 3)

- If the true proportion of voters who support Proposition *A* is $P = 0.4$, what is the probability that a sample of size 200 yields a sample proportion between 0.40 and 0.45?
- i.e.: **if $P = 0.4$ and $n = 200$, what is**

$$P\left(0.40 \le \hat{p} \le 0.45\right)?$$

Example 2 (2 of 3)

- **if P = 0.4 and n = 200, what is**

$$\boldsymbol{P\left(0.40 \le \hat{p} \le 0.45\right)?}$$

Find $\sigma_{\hat{p}}$: $\sigma_{\hat{p}} = \sqrt{\dfrac{P(1-P)}{n}} = \sqrt{\dfrac{.4(1-.4)}{200}} = .03464$

Convert to standard normal: $P(.40 \le \hat{p} \le .45) = P\left(\dfrac{.40-.40}{.03464} \le Z \le \dfrac{.45-.40}{.03464}\right)$

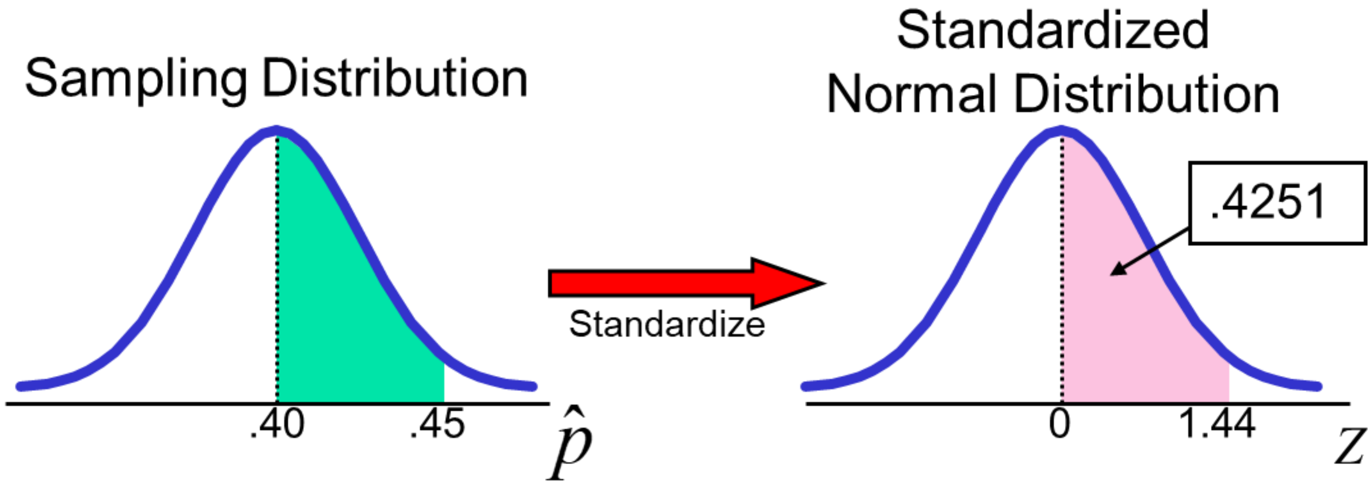
$$= P(0 \le Z \le 1.44)$$

Example 2 (3 of 3)

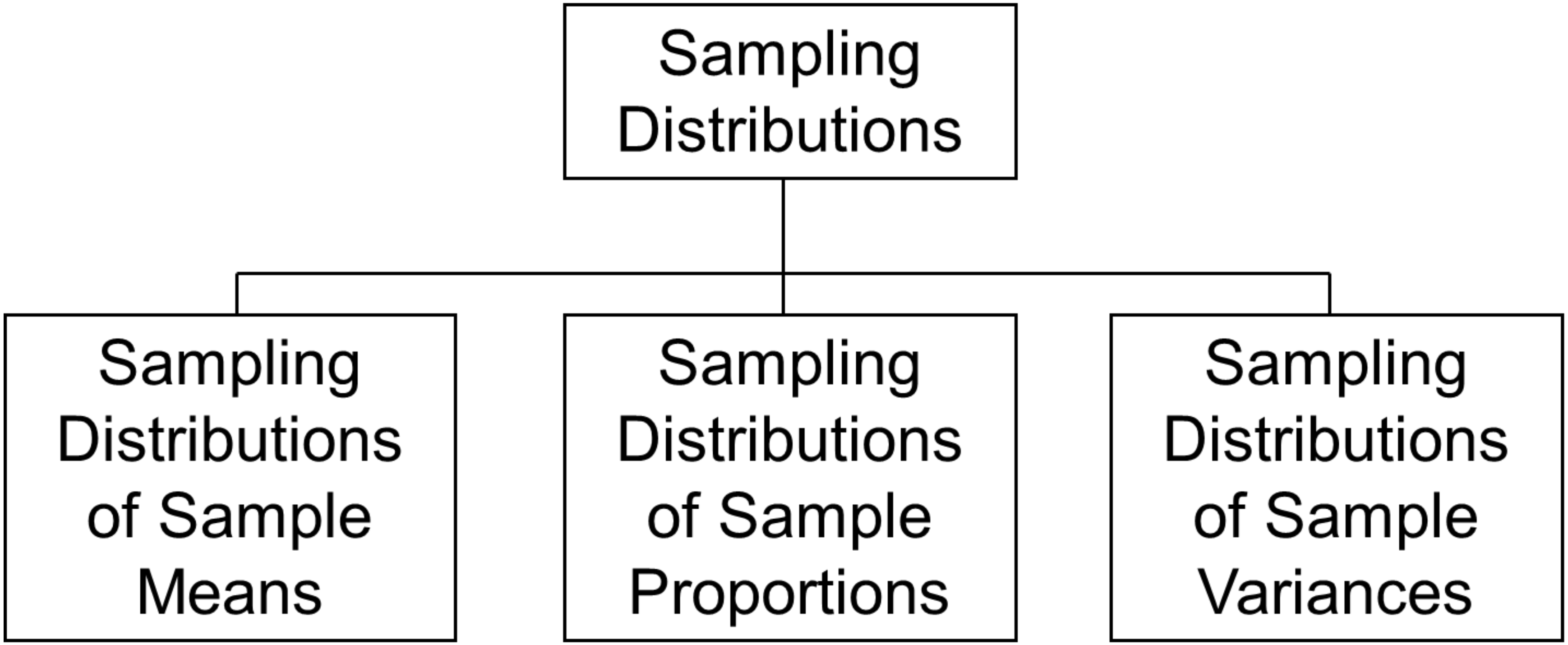
- **if P = 0.4 and n = 200, what is**

  $$P\left(0.40 \le \hat{p} \le 0.45\right)?$$

Use standard normal table: $P\left(0 \le Z \le 1.44\right) = \boxed{.4251}$



Section 6.4 Sampling Distributions of Sample Variances



Sample Variance

- Let $x_1, x_2, \ldots, x_n$ be a random sample from a population. The sample variance is

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

- the square root of the sample variance is called the sample standard deviation

- the sample variance is different for different random samples from the same population

Sampling Distribution of Sample Variances

- The sampling distribution of s^2 has mean σ^2

$$E(s^2) = \sigma^2$$

- If the population distribution is normal, then

$$Var(s^2) = \frac{2\sigma^4}{n-1}$$

Chi-Square Distribution of Sample and Population Variances

- If the population distribution is normal then

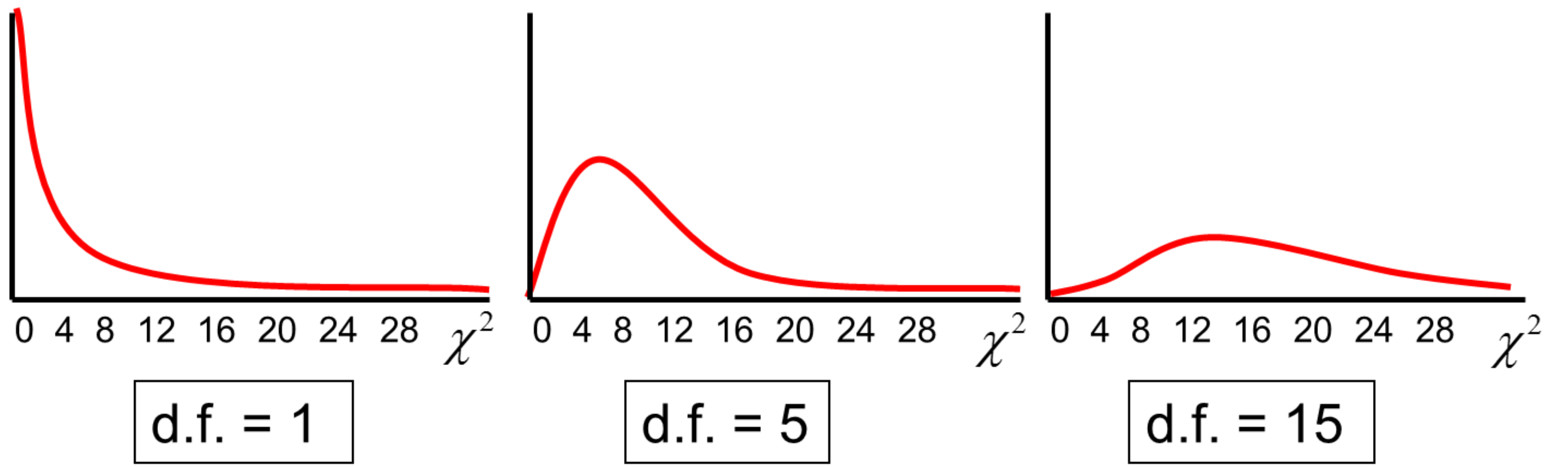
$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

has a chi-square (χ^2) distribution

with n − 1 degrees of freedom

The Chi-Square Distribution

- The chi-square distribution is a family of distributions, depending on degrees of freedom:
- d.f. = $n - 1$



- Text Appendix Table 7 contains chi-square probabilities

Degrees of Freedom (df)

Idea: Number of observations that are free to vary after sample mean has been calculated

Example: Suppose the mean of 3 numbers is 8.0

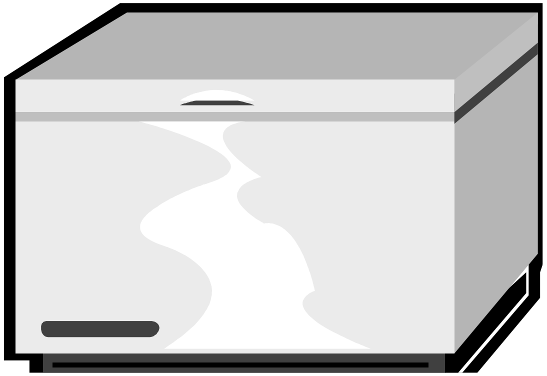
Let $X_1 = 7$ Let $X_2 = 8$ What is $\boldsymbol{X_3}$?	⇒	If the mean of these three values is 8.0, then X_3 must be 9 (i.e., X_3 is not free to vary)

Here, n = 3, so degrees of freedom = n − 1 = 3 − 1 = 2

(2 values can be any numbers, but the third is not free to vary for a given mean)

Chi-Square Example (1 of 2)

- A commercial freezer must hold a selected temperature with little variation. Specifications call for a standard deviation of no more than 4 degrees (a variance of 16 degrees^2).
- A sample of 14 freezers is to be tested
- What is the upper limit (*K*) for the sample variance such that the probability of exceeding this limit, given that the population standard deviation is 4, is less than 0.05?

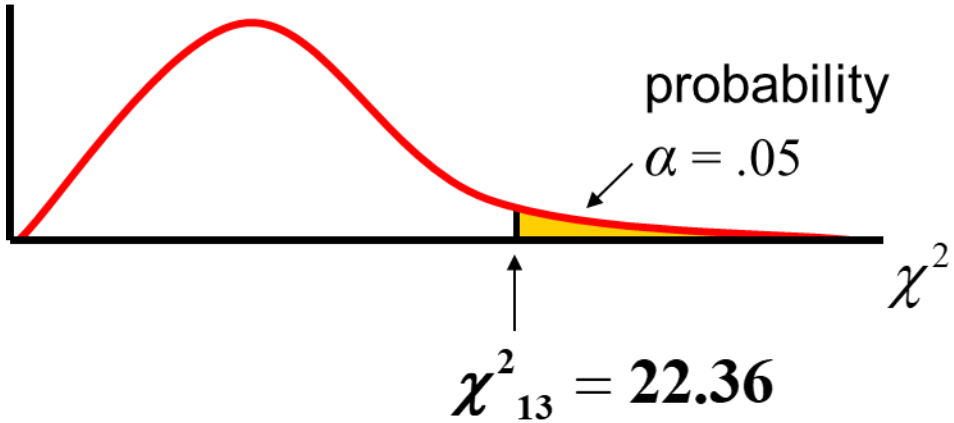


Finding the Chi-Square Value

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$ Is chi-square distributed with $(n-1) = 13$ degrees of freedom

- Use the the chi-square distribution with area 0.05 in the upper tail:

$$\chi^2_{13} = 22.36 \; (\alpha = .05 \text{ and } 14-1=13 \text{ d.f.})$$



Chi-Square Example (2 of 2)

$$\chi^2_{13} = 22.36 \ (\alpha = .05 \text{ and } 14-1=13 \text{ d.f.})$$

So: $$P(s^2 > K) = P\left(\frac{(n-1)s^2}{16} > \chi^2_{13}\right) = 0.05$$

or $$\frac{(n-1)K}{16} = 22.36$$ (where n = 14)

so $$K = \frac{(22.36)(16)}{(14-1)} = 27.52$$

If s^2 from the sample of size n = 14 is greater than 27.52, there is strong evidence to suggest the population variance exceeds 16.

Chapter Summary

- Introduced sampling distributions
- Described the sampling distribution of sample means
  - For normal populations
  - Using the Central Limit Theorem
- Described the sampling distribution of sample proportions
- Introduced the chi-square distribution
- Examined sampling distributions for sample variances
- Calculated probabilities using sampling distributions